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Inductive Post Arrays in Rectangular Waveguide

By T. A. ABELE

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Previous attempts, based on mode-matching techniques, to obtain precise data for the equivalent circuit of inductive post arrays in rectangular waveguide have consistently failed due to convergence problems. A different formulation is presented for symmetrical post arrays, which is shown to be free from this defect.

I. INTRODUCTION

Waveguide band-pass structures employing cascades of inductive posts have been built for many years. They usually contain a symmetrical arrangement of posts in each cross-section, mostly one, two or three posts. The latter arrangement, for instance, is a favorite for $\lambda/4$-coupled filters, since it strongly reduces higher order mode interaction. This allows $\lambda/4$ spacings to be used instead of the $3\lambda/4$ spacings required for single post filters, thus leading to shorter filters.

In the past all of these structures had to be designed on the basis of measured data for the equivalent circuit of the cross-sectional post arrangement, because the available theoretical calculations[1,2,8,9] are not sufficiently accurate. The obvious problem with measured data is, of course, that two errors are introduced, whose magnitudes are only poorly known: dimensional tolerances of the sample to be measured and errors in the measurement itself.

Previous attempts to obtain theoretical data based on mode-matching techniques have consistently failed due to the convergence problem typically associated with taking a finite number of unknowns out of two sets of infinitely many unknowns. This paper presents a formulation which leads to only one set of infinitely many unknowns in the case of single or double posts. It may thus be expected that, when a finite number of these is taken, no convergence problem will be encountered. One may also speculate that this will continue to be true for arrays involving three or more posts, although in these cases more than one set of infinitely many unknowns is encountered again.

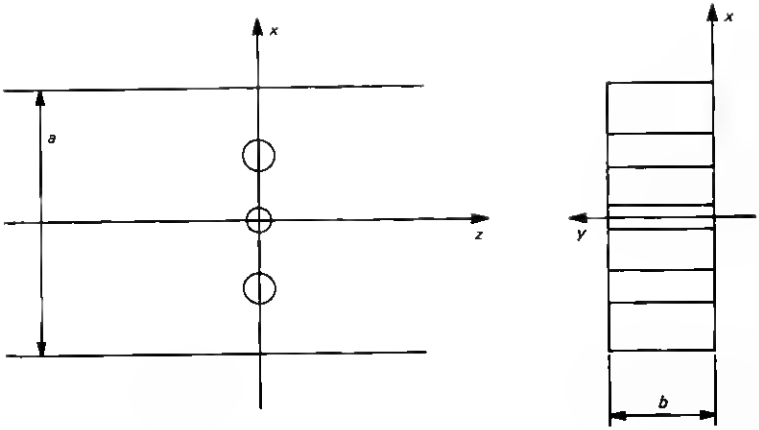


Fig. 1—Post array.

II. CONFIGURATION

We wish to determine the equivalent circuit of the array in Fig. 1 in the plane $z = 0$.

The posts are circular. They are numbered consecutively from $\mu = -M$ to $\mu = M$ with $\mu = 0$ designating the center post. The array is symmetrical with respect to the plane $z = 0$ and the plane $x = 0$. The center post may or may not be present. Each post μ has a diameter d_μ and a coordinate $x = p_\mu$ of its axis. Only dominant (TE_{10}) mode propagation is assumed. The surfaces shall be perfectly conducting.

The electric field will be calculated as the superposition of two fields; the field which exists without the posts, the unperturbed field, and the field generated by the surface currents on the posts, the perturbation field. The surface currents, or rather the coefficients of their Fourier series, are treated as unknowns, which are subsequently determined in such a way that the tangential electric field vanishes on the surface of the posts. As usual, only two special cases of excitation are studied, even and odd, since this suffices to determine the equivalent circuit.

III. UNPERTURBED FIELD

We set

$$E_{y\,\text{even}} = (e^{j\beta_g z} + e^{-j\beta_g z}) \cos \frac{\pi x}{a} \tag{1a}$$

$$E_{y\,\text{odd}} = (e^{j\beta_g z} - e^{-j\beta_g z}) \cos \frac{\pi x}{a} \tag{1b}$$

with

$$\beta_g = \left| \sqrt{\beta^2 - \frac{\pi^2}{a^2}} \right| \tag{2}$$

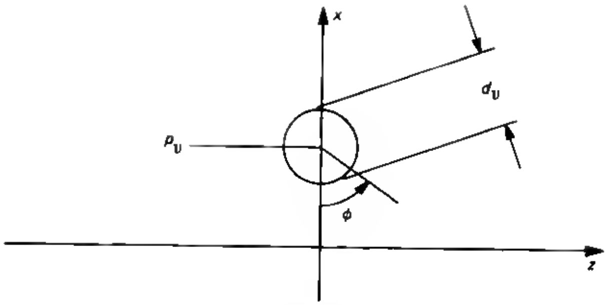


Fig. 2—Post surface.

where (dominant mode assumption)

$$\beta = \frac{2\pi}{\lambda} > \frac{\pi}{a} \tag{3}$$

λ is the wavelength in our medium. Obviously the fields in Eqs. (1) fulfill the boundary conditions everywhere except on the post surfaces.

For later use we wish to develop these fields into Fourier series on the surface of a post ν located at p_ν and of diameter d_ν.

From Fig. 2 we see that the post surface has the coordinates

$$x = p_\nu - \frac{1}{2} d_\nu \cos \phi \tag{4a}$$

$$z = \frac{1}{2} d_\nu \sin \phi \tag{4b}$$

Introduction of these expressions into Eqs. (1) and use of the well-known expansion of $e^{jz \sin\theta}$ (Ref. 5, p. 22), results in

$$E_{y\text{ even}} = \sum_{n=-\infty}^{\infty} J_n\left(\frac{1}{2}\beta d_\nu\right) e^{jn\phi} \times \left[\cos\left(\frac{\pi}{a} p_\nu - n\phi_0\right) + (-1)^n \cos\left(\frac{\pi}{a} p_\nu + n\phi_0\right)\right] \tag{5a}$$

$$E_{y\text{ odd}} = \sum_{n=-\infty}^{\infty} J_n\left(\frac{1}{2}\beta d_\nu\right) e^{jn\phi} \times \left[\cos\left(\frac{\pi}{a} p_\nu - n\phi_0\right) - (-1)^n \cos\left(\frac{\pi}{a} p_\nu + n\phi_0\right)\right] \tag{5b}$$

where

$$e^{j\phi_0} = \frac{1}{\beta}\left(\beta_g + j\frac{\pi}{a}\right) \tag{6}$$

IV. PERTURBATION FIELD

To determine the field generated by the current distributions on the posts we observe first that all of these currents are independent of y and in the direction of y. Secondly, the effect of the broad waveguide walls can be replaced, making use of the common imaging technique, by assuming that all posts are infinitely long in both y directions, and, again, have current distributions which are independent of y and in the direction of y. Thirdly, by employing the same imaging technique once more, we can replace the effect of the narrow walls by periodically repeating the array of infinitely long posts in both x directions with post locations and current distributions, which are consecutive mirror images of each other. To determine the perturbation field, we can then simply sum up the fields generated by these infinitely many and infinitely long posts, without having to worry about the boundary conditions on the waveguide walls, since they are automatically fulfilled.

From basic electromagnetic theory we get for the electric field generated by a current filament stretching in y direction from $-\infty$ to ∞, located at x_0, z_0, and of strength I_y, only the following component in y-direction

$$E_y = -\frac{j\omega\mu}{4\pi} I_y \int_{-\infty}^{\infty} \frac{e^{-j\beta\left|\sqrt{(x-x_0)^2+y_0^2+(z-z_0)^2}\right|}}{\left|\sqrt{(x-x_0)^2+y_0^2+(z-z_0)^2}\right|}\, dy_0$$

$$= -\frac{\omega\mu}{4} I_y H_0^{(2)}\left(\beta\left|\sqrt{(x-x_0)^2+(z-z_0)^2}\right|\right) \tag{7}$$

The latter transformation may be found in Ref. 3, p. 27. μ is the permeability of the medium.

Making use of the symmetry of our structure and summing over all currents on all post surfaces we obtain

$$E_y = -\frac{\omega\mu}{4} \sum_{\mu=-M}^{M} \int_0^{2\pi} I_\mu(\psi) \sum_{k=-\infty}^{\infty} (-1)^k H_0^{(2)}$$
$$\left(\beta\left|\sqrt{\left(z-\frac{1}{2}d_\mu \sin\psi\right)^2+\left(x-p_\mu+ka+\frac{1}{2}d_\mu\cos\psi\right)^2}\right|\right) d\psi \tag{8}$$

Figure 3 explains the quantities $I_\mu(\psi)$, d_μ, p_μ and the coordinates used.

With this expression for the perturbation field we will do two things. First we will determine its value at a large distance to obtain expressions for the elements of the dominant-mode equivalent circuit. Secondly, we will evaluate it on the post surfaces in order to be able to come up with an expression for the boundary condition for the tangential electric field there.

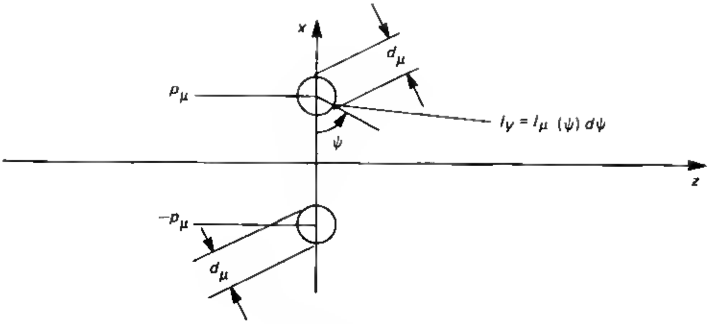


Fig. 3—Post.

V. PERTURBATION FIELD AT A LARGE DISTANCE

If we write

$$B = \frac{1}{a}\left|z - \frac{1}{2} d_\mu \sin\psi\right|$$

$$C = \frac{1}{a}\left(x - p_\mu + \frac{1}{2} d_\mu \cos\,\Big|\right)$$

$$A = \beta a$$

in Eq. (8), we can apply Eq. (34) from Appendix A to Eq. (8), which results in

$$E_y = -\omega\mu \sum_{\mu=-M}^{M} \int_0^{2\pi} I_\mu(\psi)$$

$$\left\{\sum_{k=1}^{l} \frac{\exp - j\left|\sqrt{\beta^2 - \frac{(2k-1)^2\pi^2}{a^2}}\right| \left|z - \frac{1}{2} d_\mu \sin\psi\right|}{a\left|\sqrt{\beta^2 - \frac{(2k-1)^2\pi^2}{a^2}}\right|}\right.$$

$$\times \cos\left[\frac{(2k-1)\pi}{a}\left(x - p_\mu + \frac{1}{2} d_\mu \cos\psi\right)\right]$$

$$+ j \sum_{k=l+1}^{\infty} \frac{\exp - \left|\sqrt{\frac{(2k-1)^2\pi^2}{a^2} - \beta^2}\right| \left|z - \frac{1}{2} d_\mu \sin\psi\right|}{a\left|\sqrt{\frac{(2k-1)^2\pi^2}{a^2} - \beta^2}\right|}$$

$$\left.\times \cos\left[\frac{(2k-1)\pi}{a}\left(x - p_\mu + \frac{1}{2} d_\mu \cos\psi\right)\right]\right\} d\psi \quad (9)$$

with

$$|z| > \frac{1}{2} d_\mu, \quad \frac{(2l-1)\pi}{a} < \beta < \frac{(2l+1)\pi}{a}$$

The second of the two sums over y obviously represents the evanescent modes in the rectangular waveguide and, therefore, vanishes for large $|z|$. In accordance with our assumptions we have

$$\frac{\pi}{a} < \beta < \frac{3\pi}{a} \tag{10}$$

which means that $l = 1$ in Eq. (9). We therefore find from Eq. (9) for

$$\frac{\pi z}{a} \gg 1$$

$$E_y = -\frac{\omega\mu}{\beta_g a} \sum_{\mu=-M}^{M} \int_0^{2\pi} I_\mu(\psi) e^{-j\beta_g(z-\frac{1}{2}d_\mu \sin\psi)} \times \cos\left[\frac{\pi}{a}\left(x - p_\mu + \frac{1}{2} d_\mu \cos\psi\right)\right] d\psi$$

$$= -\frac{\omega\mu}{2\beta_g a} \sum_{\mu=-M}^{M} \int_0^{2\pi} I_\mu(\psi)\, [e^{-j\beta_g z + j\pi(x-p_\mu)/a} e^{j\frac{1}{2}\beta d_\mu \sin(\psi+\phi_0)} + e^{-j\beta_g z - j\pi(x-p_\mu)/a} e^{j\frac{1}{2}\beta d_\mu \sin(\psi-\phi_0)}]\, d\psi \tag{11}$$

where ϕ_0 is again defined by Eq. (6). Using once more the expansion already used in Eqs. (5) we obtain

$$E_y = -\frac{\omega\mu}{\beta_g a} e^{-j\beta_g z} \sum_{\mu=-M}^{M} \sum_{m=-\infty}^{\infty} J_m\left(\frac{1}{2}\beta d_\mu\right) \times \cos\left[\frac{\pi}{a}(x - p_\mu) + m\phi_0\right] \int_0^{2\pi} I_\mu(\psi) e^{jm\psi} d\psi \tag{12}$$

The inversion of the order of integration and summation employed here presents no difficulty.

Physical considerations tell us that the currents $I_\mu(\psi)$ can be developed into a Fourier series. We write in the usual manner

$$I_\mu(\psi) = \sum_{m=-\infty}^{\infty} c_{\mu,m} e^{jm\psi} \tag{13a}$$

where, of course,

$$c_{\mu,m} = \frac{1}{2\pi} \int_0^{2\pi} I_\mu(\psi) e^{-jm\psi} d\psi \tag{13b}$$

Eq. (13b), when combined with Eq. (12), results in

$$E_y = -\frac{2\pi\omega\mu}{\beta_g a} e^{-j\beta_g z} \sum_{\mu=-M}^{M} \sum_{m=-\infty}^{\infty} (-1)^m c_{\mu,m} J_m\left(\frac{1}{2}\beta d_\mu\right) \times \cos\left[\frac{\pi}{a}(x - p_\mu) - m\phi_0\right] \tag{14}$$

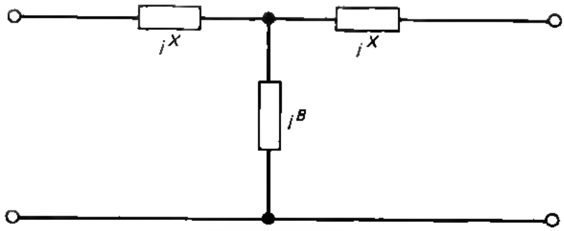


Fig. 4—Equivalent circuit.

For $x = 0$ (center of the guide) this results in

$$E_y = -\frac{2\pi\omega\mu}{\beta_g a} e^{-j\beta_g z} \sum_{\mu=-M}^{M} \sum_{m=-\infty}^{\infty} (-1)^m c_{\mu,m} J_m \left(\frac{1}{2}\beta d_\mu\right) \times \cos\left(\frac{\pi}{a} p_\mu + m\phi_0\right) \quad (15)$$

Combining this result with Eqs. (1) for $x = 0$ we find for the total (unperturbed plus perturbation) field for $\pi z/a \gg 1$

$$E_{y_{\text{odd}}^{\text{even}}} = e^{j\beta_g z} \pm e^{-j\beta_g z} - \frac{2\pi\omega\mu}{\beta_g a} e^{-j\beta_g z} \times \sum_{\mu=-M}^{M} \sum_{m=-\infty}^{\infty} (-1)^m c_{\mu,m_{\text{odd}}^{\text{even}}} J_m \left(\frac{1}{2}\beta d_\mu\right) \cos\left(\frac{\pi}{a} p_\mu + m\phi_0\right) \quad (16)$$

$c_{\mu,m_{\text{odd}}^{\text{even}}}$ is written here to distinguish between the values of $c_{\mu,m}$ for even and odd excitation. For the equivalent circuit of Fig. 4, which is valid for the plane $z = 0$, we obtain from Eq. (16) the reflection coefficient

$$\rho_{\text{odd}}^{\text{even}} = \pm 1 - \frac{2\pi\omega\mu}{\beta_g a} \times \sum_{\mu=-M}^{M} \sum_{m=-\infty}^{\infty} (-1)^m c_{\mu,m_{\text{odd}}^{\text{even}}} J_m \left(\frac{1}{2}\beta d_\mu\right) \cos\left(\frac{\pi}{a} p_\mu + m\phi_0\right) \quad (17)$$

with

$$\rho_{\text{even}} = \frac{jX + \dfrac{2}{jB} - 1}{jX + \dfrac{2}{jB} + 1} \quad (18a)$$

$$\rho_{\text{odd}} = \frac{jX - 1}{jX + 1} \quad (18b)$$

These equations permit the calculation of X and B once the values of $c_{\mu,m_{\text{odd}}^{\text{even}}}$ are known.

We note, that because of the structural symmetry with respect to $x = 0$

$$c_{-\mu,m} = (-1)^m c_{\mu,-m} \tag{19}$$

for $\mu = 0, \pm 1, \pm 2 \ldots$ Also, since we have

$$I_{\mu_{\text{odd}}^{\text{even}}}(\psi) = \pm I_{\mu_{\text{odd}}^{\text{even}}}(2\pi - \psi) \tag{20}$$

it follows that

$$c_{\mu,-m_{\text{odd}}^{\text{even}}} = \pm c_{\mu,m_{\text{odd}}^{\text{even}}} \tag{21}$$

Eqs. (19) and (21) permit reduction of $c_{\mu,m}$ for negative values of μ and/or m to those with positive values.

VI. PERTURBATION FIELD ON POST SURFACES

Referring once more to Fig. 2 we get for the perturbation field on the surface of a post ν from Eqs. (8) and (4)

$$\begin{aligned} E_y = -\frac{\omega\mu}{4} \sum_{\mu=-M}^{M} \int_0^{2\pi} I_\mu(\psi) \sum_{k=-\infty}^{\infty} (-1)^k \\ \times H_0^{(2)} \left(\beta \left| \left(\frac{1}{2} d_\nu \sin\phi - \frac{1}{2} d_\mu \sin\psi\right)^2 \right.\right. \\ \left.\left. + \left(p_\nu - p_\mu + ka - \frac{1}{2} d_\nu \cos\phi + \frac{1}{2} d_\mu \cos\psi\right)^2 \right|^{1/2}\right) d\psi \end{aligned} \tag{22}$$

We wish to write for this a double Fourier series with ϕ and ψ as independent variables. This can be done with tbe aid of the so-called "addition" theorem (Ref. 5, p. 361) if we impose the condition that the posts do not penetrate or touch each other or tbe narrow walls of the waveguide. We obtain

$$\begin{aligned} E_y = -\frac{\omega\mu}{4} \sum_{\mu=-M}^{M} \int_0^{2\pi} I_\mu(\psi) \\ \times \Bigg[\sum_{\substack{k=0 \\ ka+p_\nu-p_\mu>0}}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{m+k} J_m\left(\frac{1}{2}\beta d_\mu\right) J_n\left(\frac{1}{2}\beta d_\nu\right) \\ \times H_{n+m}^{(2)} \{\beta(p_\nu - p_\mu + ka)\} e^{j(n\phi + m\psi)} \\ + \sum_{\substack{k=0 \\ ka-p_\nu+p_\mu>0}}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_m\left(\frac{1}{2}\beta d_\mu\right) J_n\left(\frac{1}{2}\beta d_\nu\right) \\ \times H_{n+m}^{(2)} \{\beta(p_\mu - p_\nu + ka)\} e^{j(n\phi + m\psi)} (-1)^{n+k} \\ + \lim_{\substack{\kappa\to 1 \\ \mu=\nu \text{ only}}} \sum_{n=-\infty}^{\infty} J_n\left(\frac{1}{2}\beta d_\nu\right) H_n^{(2)}\left(\frac{1}{2}\kappa\beta d_\nu\right) e^{jn(\phi-\psi)} \Bigg] d\psi \end{aligned} \tag{23}$$

Based on physical reasoning (summing contributions of current filaments in different order, integrating around each post before summing) we now exchange the order of summations and integration in Eq. (23) and carry out the integration. This leads to

$$E_y = -\frac{\omega\mu\pi}{2}\sum_{n=-\infty}^{\infty} J_n\left(\frac{1}{2}\beta d_\nu\right)e^{jn\phi}\left[\sum_{\mu=-M}^{M}\sum_{m=-\infty}^{\infty} J_m\left(\frac{1}{2}\beta d_\mu\right)c_{\mu,m}\right.$$

$$\times \sum_{\substack{k=-\infty \\ p_\mu - p_\nu + ka \neq 0}}^{\infty} (-1)^k H_{m-n}^{(2)}(\beta|p_\mu - p_\nu + ka|)[\operatorname{sgn}(p_\mu - p_\nu + ka)]^{n+m}$$

$$\left. + \lim_{\kappa\to 1} H_n^{(2)}\left(\frac{1}{2}\kappa\beta d_\nu\right)c_{\nu,n}\right] \quad (24)$$

With the abbreviation

$$\sum_{\substack{k=-\infty \\ A+kB\neq 0}}^{\infty} (-1)^k H_m^{(2)}(|A+kB|)\left[\operatorname{sgn}\left(\frac{A}{B}+k\right)\right]^m$$

$$= f_m(A,B) = (-1)^m f_{-m}(A,B) = (-1)^m f_m(-A,B) \quad (25)$$

we can write this as

$$E_y = -\frac{\omega\mu\pi}{2}\sum_{n=-\infty}^{\infty} J_n\left(\frac{1}{2}\beta d_\nu\right) e^{jn\phi}$$

$$\times\left[\sum_{\mu=-M}^{m}\sum_{m=-\infty}^{\infty} J_m\left(\frac{1}{2}\beta d_\mu\right) c_{\mu,m} f_{m-n}\{\beta(p_\mu - p_\nu),\beta a\}\right.$$

$$\left. + \lim_{\kappa\to 1} H_n^{(2)}\left(\frac{1}{2}\kappa\beta d_\nu\right)c_{\nu,n}\right] \quad (26)$$

If we take this result for the perturbation field, add it to the incident field Eqs. (5) and impose the condition $E_y = 0$ on the surface of the post, we get (letting $\kappa \to 1$)

$$\sum_{n=-\infty}^{\infty} J_n\left(\frac{1}{2}\beta d_\nu\right) e^{jn\phi}\left[\cos\left(\frac{\pi}{a}p_\nu - n\phi_0\right) \pm (-1)^n \cos\left(\frac{\pi}{a}p_\nu + n\phi_0\right)\right]$$

$$= \frac{\omega\mu\pi}{2}\sum_{n=-\infty}^{\infty} J_n\left(\frac{1}{2}\beta d_\nu\right) e^{jn\phi}\left[\sum_{\mu=-M}^{M}\sum_{m=-\infty}^{\infty} J_m\left(\frac{1}{2}\beta d_\mu\right)\right.$$

$$\left.\times\, c_{\mu,m_{\text{odd}}^{\text{even}}} f_{m-n}\{\beta(p_\mu - p_\nu), \beta a\} + H_n^{(2)}\left(\frac{1}{2}\beta d_\nu\right) c_{\nu,n_{\text{odd}}^{\text{even}}}\right] \quad (27)$$

Because of the uniqueness of Fourier series (Ref. 4, p. 186), this results in

$$\cos\left(\frac{\pi}{a}p_\nu - n\phi_0\right) \pm (-1)^n \cos\left(\frac{\pi}{a}p_\nu + n\phi_0\right)$$

$$= \frac{\omega\mu\pi}{2}\left[\sum_{\mu=-M}^{M}\sum_{m=-\infty}^{\infty} J_m\left(\frac{1}{2}\beta d_\mu\right) c_{\mu,m_{\text{odd}}^{\text{even}}} f_{m-n}\{\beta(p_\mu - p_\nu),\beta a\}\right.$$

$$\left. + H_n^{(2)}\left(\frac{1}{2}\beta d_\nu\right) c_{\nu,n_{\text{odd}}^{\text{even}}}\right] \quad (28)$$

This equation holds for $n = 0, 1, 2 \ldots$. It expresses the boundary condition on the surface of post ν for even and odd excitation. If applied to all posts $\nu = 0, \pm 1, \pm 2 \ldots \pm M$, it expresses the boundary condition on all posts. However, because of the symmetry involved, only $\nu = 0, 1, 2 \ldots M$ are needed. As before, Eqs. (19) and (21) permit reduction of $c_{\mu,m}$ for negative values of μ and/or m to those with positive values. In summary we can say that Eq. (28), if applied for $n = 0, 1, 2, \ldots$ and $\nu = 0, 1, 2 \ldots M$, will allow us to compute all of the unknown coefficients $c_{\nu,m}$. In turn, Eq. (17) will then allow us to compute the elements of the equivalent circuit, which means that our problem is solved.

Appendix A and Appendix B provide expressions suitable for the computation of $f_m(A,B)$ in Eq. (28). These alternate expressions are essential, because the defining series Eq. (25) converges very slowly, as the magnitude of $H_m^{(2)}(z)$ decreases only with $|z^{-1/2}|$ for large z. The derivation of these expressions constitutes the most difficult and laborious part of this analysis. For convenience the results are repeated below in the form most appropriate for Eq. (28). From Eqs. (40) and Eq. (41),

$$f_m(\beta p, \beta a) = \frac{4}{\pi} \tan \phi_0 \cos\left(\frac{\pi p}{a} - \frac{m\pi}{2}\right) e^{jm(\phi_0 - \pi/2)}$$

$$+ j\frac{1}{\pi} \sum_{n=3,5\ldots} \frac{\frac{2\lambda}{a}\cos\left(\frac{n\pi p}{a} - \frac{m\pi}{2}\right)}{\left|\sqrt{\left(\frac{n\lambda}{2a}\right)^2 - 1}\right| \left[\frac{n\lambda}{2a} + \left|\sqrt{\left(\frac{n\lambda}{2a}\right)^2 - 1}\right|\right]^m}$$

$$+ j\frac{1}{\pi} \sum_{n=0}^{<\frac{m}{2}} \frac{(m-n-1)!}{n!(m-2n-1)!}$$

$$\times \left[\frac{\lambda}{a \sin \frac{\pi p}{a}}\right]^{m-2n} h_{m-2n-1}\left(\cos\frac{\pi p}{a}\right) \tag{29a}$$

$$f_{2m-1}(\beta N a, \beta a) = 0 \tag{29b}$$

$$f_{2m}(\beta N a, \beta a) = \frac{4}{\pi} \tan \phi_0 (-1)^{N+m} e^{j2m(\phi_0 - \pi/2)}$$

$$+ j\frac{1}{\pi} \sum_{n=3,5\ldots} \frac{\frac{2\lambda}{a}(-1)^{N+m}}{\left|\sqrt{\left(\frac{n\lambda}{2a}\right)^2 - 1}\right| \left[\frac{n\lambda}{2a} + \left|\sqrt{\left(\frac{n\lambda}{2a}\right)^2 - 1}\right|\right]^{2m}}$$

$$+ j\frac{1}{\pi} \sum_{n=0}^{m} \frac{2(-1)^{N+n}(m+n-1)!(2^{2n-1}-1)B_{2n}}{(m-n)!(2n)!} \left(\frac{\lambda}{a}\right)^{2n} \tag{29c}$$

From Eq. (34e) and Eq. (34f)

$$f_0(\beta p, \beta a) = \frac{4}{\pi} \frac{\cos \dfrac{\pi p}{a}}{\cos \phi_0} e^{j(\phi_0 - \pi/2)} + j \frac{1}{\pi} \ln \left(\cot^2 \frac{\pi p}{2a} \right)$$

$$+ j \frac{1}{\pi} \sum_{n=3,5...} \frac{4 \cos \dfrac{n\pi p}{a}}{\left| \sqrt{\left(\dfrac{n\lambda}{2a}\right)^2 - 1} \right| \left[\dfrac{n\lambda}{2a} + \left| \sqrt{\left(\dfrac{n\lambda}{2a}\right)^2 - 1} \right| \right] n} \tag{29d}$$

$$f_0(\beta Na, \beta a) = \frac{4}{\pi} \frac{(-1)^N}{\cos \phi_0} e^{j(\phi_0 - \pi/2)} - (-1)^N + j \frac{2}{\pi} \left(C + \ln \frac{2a}{\lambda} \right) (-1)^N$$

$$+ j \frac{1}{\pi} \sum_{n=3,5...} \frac{4(-1)^N}{\left| \sqrt{\left(\dfrac{n\lambda}{2a}\right)^2 - 1} \right| \left[\dfrac{n\lambda}{2a} + \left| \sqrt{\left(\dfrac{n\lambda}{2a}\right)^2 - 1} \right| \right] n} \tag{29e}$$

These expressions are valid if

$$2a > \lambda > \frac{2a}{3}$$

$$p \neq 0, \pm a, \pm 2a \ldots$$

$$m = 1, 2, 3, \ldots$$

$$N = 0, \pm 1, \pm 2 \ldots$$

The polynomials $h_m(x)$ are defined in Appendix B, Eq. (42). B_n is the nth Bernoullian number and C is Euler's constant.

VII. NUMERICAL RESULTS

A series of calculations was made to investigate the question of convergence and to ascertain that the rather involved analysis is error-free. To this end the reactances $X - 2/B$ and $-X$ [Eqs. (18)] were calculated for the cases of single, double, and triple posts with $\lambda/a = 1.2$, $p_1/a = 0.25$ and $\beta d_0 = \beta d_1 = 0.2$ and 0.4, employing increasing numbers of variables and equations. Furthermore, the computed results were compared with measured data where such data were available. Lacking a full-fledged computer program the calculations were carried out with the aid of a programmable desk calculator, except for the matrix inversion, for which a general-purpose computer program was used.

Table I summarizes the results of this work. The first observation that can be made is that, as expected, the convergence obtained for single and double posts is excellent. Three terms in the Fourier series for the post currents is all that is needed to obtain six place accuracy for the reac-

Table I

Number of posts	$\frac{p_1}{a}$	βd_0	βd_1	$n_{\max}$	$X - \frac{2}{B}$	$-X$
1	—	0.2	—	0	1.121835970	—
				1	—	.009450748398
				2	1.121835438	—
				3	—	.009450749381
				4	1.121835438	—
				Meas.	1.12	.010
		0.4		0	.6546985053	—
				1	—	.03659710000
				2	.6546719818	—
				3	—	.03659716655
				4	.6546719813	—
				Meas.	.655	.037
2	0.25	—	0.2	0	1.121835969	—
				1	1.100680471	.009467748389
				2	1.100679708	.009467748828
				3	1.100679707	.009467799814
				4	1.100679707	.009467799806
			0.4	0	.6546985046	—
				1	.6071886432	.03659710003
				2	.6071426563	.03685942441
				3	.6071423836	.03685949132
				4	.6071423830	.03685949138
3	0.25	0.2	0.2	0	.2599117670	—
				1	.2578995041	.01871192452
				2	.2578489230	.01872800993
				3	.2578488656	.01872801335
				4	.2578488652	.01872801340
				Meas.	.255	.020
		0.4	0.4	0	.02634303527	—
				1	.02329626388	.07043107151
				2	.02238859942	.07063929772
				3	.02238444183	.07063993488
				4	.02238437265	.07063993658
				Meas.	.0265	.074

tances, which is more than enough for any technical application. The second observation is that, as was hoped, excellent convergence continues to exist for triple posts, even though in that case two sets of infinitely many unknowns are encountered instead of just one. Even for posts with susceptance values as high as $B = 20$ no more than four terms in the Fourier series are needed to obtain five-place accuracy. Presumably the analysis will converge even for four or more posts, but these arrangements are of little technical interest and thus probably not worth investigating. Finally, when comparing the computed values with measured data obtained with the aid of a very precise, computer-operated transmission measurement set,[10] sufficient agreement is found to ascertain that the analysis is free from any fundamental error.

APPENDIX A

We study the following series

$$f(z, B, C, t) = \sum_{n=-\infty}^{\infty} \frac{e^{-B\sqrt{[(2n-1)\pi+t]^2+z^2}} \cos\{[(2n-1)\pi + t]C\}}{\sqrt{[(2n-1)\pi + t]^2 + z^2}} \tag{30}$$

with t as a real variable, $B \geq 0$ and real, C real, $\mathrm{Re}\{z\} > 0$, $\mathrm{Re}\{\sqrt{[(2n-1)\pi + t]^2 + z^2}\} > 0$. This function is even in t and also periodic in t with the period 2π. For reasons which will become apparent later we wish to develop it into a Fourier series in t

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kt \tag{31}$$

Without going into the fairly laborious detail the result is

$$f(z, B, C, t) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} (-1)^k K_0(z|\sqrt{B^2 + (C+k)^2}| \cos kt \tag{32}$$

for the conditions stated for Eq. (30) plus either $B > 0$ or $B = 0$ and $C \neq 0, \pm 1, \pm 2 \ldots$. Setting $t = 0$ leads to

$$2\sum_{n=1}^{\infty} \frac{e^{-B\sqrt{(2n-1)^2\pi^2+z^2}} \cos[(2n-1)C]}{\sqrt{(2n-1)^2\pi^2 + z^2}} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} (-1)^k K_0(z|\sqrt{B^2 + (C+k)^2}|) \tag{33}$$

provided $\mathrm{Re}\{z\} > 0$, $\mathrm{Re}\{\sqrt{(2n-1)^2\pi^2 + z^2}\} > 0$ and either $B > 0$ and real, C real or $B = 0$, $C \neq 0, \pm 1, \pm 2 \ldots$ and real. The validity of Eq. (43) can be extended to include $\mathrm{Re}\{z\} = 0$ by analytic continuation. In doing this the points $z = 0$ and $z = \pm j(2n-1)\pi$ obviously have to be excluded, since at these points individual terms of the sums involved are not analytic. The result of the analytic continuation is for $z \to jA$

$$\sum_{k=-\infty}^{\infty} (-1)^k H_0^{(2)}(A|\sqrt{B^2 + (C+k)^2}|) = 4j\sum_{n=1}^{\infty} \frac{e^{-B\sqrt{(2n-1)^2\pi^2 - A^2}}}{\sqrt{(2n-1)^2\pi^2 - A^2}} \cos[(2n-1)\pi C] \tag{34a}$$

with

$$A > 0, \text{ real}$$

$$A \neq \pi, 3\pi, 5\pi \ldots \tag{34b}$$

$$\arg\sqrt{(2n-1)^2\pi^2 - A^2} = 0 \text{ or } \frac{\pi}{2}$$

and either

$$B > 0, \text{ real}$$

$$C \text{ real} \tag{34c}$$

or

$$B = 0$$

$$C \neq 0, \pm 1, \pm 2 \ldots, \text{ real} \tag{34d}$$

For the latter situation [Eq. (34d)] Eq. (34a) may be rewritten in the more rapidly converging form

$$\sum_{k=-\infty}^{\infty} (-1)^k H_0^{(2)}(A|C+k|) = j\frac{1}{\pi}\ln\left(\cot^2\frac{1}{2}\pi A\right) + 4j\sum_{n=1}^{\infty} \frac{A^2\cos[(2n-1)\pi C]}{(2n-1)\pi\sqrt{(2n-1)^2\pi^2-A^2}\,[(2n-1)\pi+\sqrt{(2n-1)^2\pi^2-A^2}]} \tag{34e}$$

We also need a result corresponding to Eq. (34e) for $C = N = 0, \pm 1, \pm 2 \ldots$. We observe first that in the left hand sum the term $k = -N$ has to be excluded for obvious reasons. Furthermore it is

$$\sum_{\substack{k=-\infty \\ k\neq -n}}^{\infty} (-1)^k H_0^{(2)}(A|N+K|) = (-1)^N \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (-1)^k H_0^{(2)}(A|k|)$$

$$= 2(-1)^N \sum_{k=1}^{\infty} (-1)^k H_0^{(2)}(Ak) \tag{35}$$

An alternate expression for the last sum is known (Ref. 6, p. 333). We get

$$\sum_{\substack{k=-\infty \\ k\neq -N}}^{\infty} (-1)^k H_0^{(2)}(A|N+k|) = (-1)^N \left\{-1 + 2j\frac{1}{\pi}\left(C + \ln\frac{A}{4\pi}\right) + 4j\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{(2n-1)^2\pi^2-A^2}} - \frac{1}{2n\pi}\right]\right\}$$

$$= (-1)^N \left\{-1 + 2j\frac{1}{\pi}\left(C + \ln\frac{A}{\pi}\right) + 4j\sum_{n=1}^{\infty} \frac{A^2}{(2n-1)\pi\sqrt{(2n-1)^2\pi^2-A^2}[(2n-1)\pi+\sqrt{(2n-1)^2\pi^2-A^2}]}\right\} \tag{34f}$$

with $N = 0, \pm 1, \pm 2 \ldots$ and Eqs. (34b) in force. Note that C in this last formula is Euler's constant.

APPENDIX B

We study for $m = 1, 2 \ldots$ the series

$$f_m(z, C, t) = \sum_{n=1}^{\infty} \frac{z^m \cos\left\{\left[(2n-1)\pi + t\right]C - m\frac{\pi}{2}\right\}}{\sqrt{[(2n-1)\pi+t]^2+z^2}\,[(2n-1)\pi + t + \sqrt{[(2n-1)\pi+t]^2+z^2}]^m} \tag{36}$$

with t as a real variable with the range $-\pi \leq t \leq \pi$, C real $\mathrm{Re}\{z\} > 0$, $\mathrm{Re}\{\sqrt{[(2n-1)\pi + t]^2 + z^2}\} > 0$. Analogous to the situation in Appendix A we wish to develop $f_m(z, C, t) + f_m(z, C, -t)$ into a Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kt$$

over the range $-\pi \leq t \leq \pi$, thereby continuing it periodically beyond that range. Again omitting the fairly laborious detail we get for $C \neq 0, \pm 1, \pm 2 \ldots$

$$f_m(z, C, t) + f_m(z, C, -t)$$
$$= \frac{1}{8\pi}(-1)^{m+1}[4S_m(jCz)e^{jm\pi/2} - 8K_m(|C|z)(\operatorname{sgn} C)^m]$$
$$+ \frac{1}{4\pi}(-1)^{m+1} \sum_{k=1}^{\infty} (-1)^k [2S_m(jCz + jkz)e^{jm\pi/2}$$
$$+ 2S_m(jCz - jkz)e^{jm\pi/2} - 4K_m(|C+k|z)\{\operatorname{sgn}(C+k)\}^m$$
$$- 4K_m(|C-k|z)\{\operatorname{sgn}(C-k)\}^m] \cos kt = \frac{1}{2\pi}(-1)^m$$
$$\times \sum_{k=-\infty}^{\infty} (-1)^k [2K_m(|C+k|z)\{\operatorname{sgn}(C+k)\}^m$$
$$- e^{jm\pi/2} S_m(jCz + jkz)] \cos kt \qquad (37a)$$

and for $C = N = 0, \pm 1, \pm 2 \ldots$

$$f_m(z, N, t) + f_m(z, N, -t)$$
$$= \frac{1}{2\pi}(-1)^m \sum_{\substack{k \neq -\infty \\ k \neq -N}}^{\infty} (-1)^k [2K_m(|N+k|z)\{\operatorname{sgn}(N+k)\}^m$$
$$-e^{jm\pi/2} S_m(jNz + jkz)] \cos kt + \frac{1}{m\pi}(-1)^N \cos \frac{m\pi}{2} \cos Nt \qquad (37b)$$

In these equations $S_m(z)$ denotes Schlaefli's polynomial (Ref. 5, p. 313). For $t = 0$ this results in

$$\sum_{n=1}^{\infty} \frac{z^m \cos\left[(2n-1)\pi C - m\dfrac{\pi}{2}\right]}{\sqrt{(2n-1)^2\pi^2 + z^2}\,[(2n-1)\pi + \sqrt{(2n-1)^2\pi^2 + z^2}]^m}$$

$$= \frac{1}{4\pi}(-1)^m \sum_{k=-\infty}^{\infty} (-1)^k [2K_m(|C+k|z)\{\operatorname{sgn}(C+k)\}^m$$
$$- e^{jm\pi/2} S_m(jCz + jkz)] \qquad (38a)$$

provided $C \neq 0, \pm 1, \pm 2 \ldots$, and in

$$\sum_{n=1}^{\infty} \frac{z^m \cos\left[(2n-1)\pi N - m\frac{\pi}{2}\right]}{\sqrt{(2n-1)^2\pi^2+z^2}[(2n-1)\pi+\sqrt{(2n-1)^2\pi^2+z^2}]^m}$$
$$= \frac{1}{4\pi}(-1)^m \sum_{\substack{k=-\infty \\ k\neq -N}}^{\infty} (-1)^k[2K_m(|N+k|z)\,\{\mathrm{sgn}\,(N+k)\}^m$$
$$-e^{jm\pi/2} S_m(jNz+jkz)] + \frac{1}{2m\pi}(-1)^N \cos\frac{m\pi}{2} \quad (38b)$$

for $N = 0, \pm 1, \pm 2 \ldots$. By working on the two series with Schlaefli's polynomials the following alternative expressions are obtained

$$\sum_{n=1}^{\infty} \frac{z^m \cos\left[(2n-1)\pi C - m\frac{\pi}{2}\right]}{\sqrt{(2n-1)^2\pi^2+z^2}[(2n-1)\pi+\sqrt{(2n-1)^2\pi^2+z^2}]^m}$$
$$= \frac{1}{2\pi}(-1)^m \sum_{k=-\infty}^{\infty} (-1)^k K_m(|C+k|z)[\mathrm{sgn}\,(C+k)]^m$$
$$+ \frac{1}{4\pi} e^{jm\pi/2} \sum_{n=0}^{<m/2} \frac{(m-n-1)!}{n!(m-2n-1)!}\left(\frac{2\pi}{jz}\right)^{m-2n}$$
$$\times \left[\frac{d^{m-2n-1}}{dx^{m-2n-1}} \frac{1}{\sin x}\right]_{x=C\pi} \quad (39a)$$

$$\sum_{\substack{k=-\infty \\ k\neq -N}}^{\infty} (-1)^k K_{2\lambda-1}(|N+k|z)\,\mathrm{sgn}\,(N+k) = 0 \quad (39b)$$

$$\sum_{n=1}^{\infty} \frac{z^{2\lambda}(-1)^{N+\lambda}}{\sqrt{(2n-1)^2\pi^2+z^2}[(2n-1)\pi+\sqrt{(2n-1)^2\pi^2+z^2}]^{2\lambda}}$$
$$= \frac{1}{2\pi} \sum_{\substack{k=-\infty \\ k\neq -N}}^{\infty} (-1)^k K_{2\lambda}(|N+k|z) + \frac{1}{4\lambda\pi}(-1)^{N+\lambda}$$
$$- \frac{1}{2\pi}(-1)^{N+\lambda} \sum_{k=1}^{\lambda} \frac{(\lambda+k-1)!(2^{2k-1}-1)B_{2k}}{(\lambda-k)!(2k)!}\left(\frac{2\pi}{z}\right)^{2k} \quad (39c)$$

where, again, $C \neq 0, \pm 1, \pm 2 \ldots$, real, $N = 0, \pm 1, \pm 2 \ldots$, $m = 1, 2 \ldots$, $\lambda = 1, 2 \ldots$, $\mathrm{Re}\{z\} > 0$, $\mathrm{Re}\{\sqrt{(2n-1)^2\pi^2+z^2}\} > 0$. Following the same argumentation as in Appendix A the validity of Eqs. (39) can, by analytic continuation, be extended to include $\mathrm{Re}\{z\} = 0$ with the exception of $z = 0$ and $z = \pm j(2n-1)\pi$. The result is for $z \to jA$ with $A > 0$ and after some rearrangement

$$\sum_{k=-\infty}^{\infty} (-1)^k H_m^{(2)}(|C+k|A)[\mathrm{sgn}\,(C+k)]^m$$

$$= 4j \sum_{n=1}^{\infty} \frac{A^m \cos\left[(2n-1)\pi C - \frac{\pi}{2}\right]}{\sqrt{(2n-1)^2\pi^2 - A^2}[(2n-1)\pi + \sqrt{(2n-1)^2\pi^2 - A^2}]^m}$$

$$- j(-1)^m \frac{1}{\pi} \sum_{n=0}^{\leq \frac{m}{2}} \frac{(m-n-1)!}{n!(n-2n-1)!} \left(\frac{2\pi}{A}\right)^{m-2n}$$

$$\times \left[\frac{d^{m-2n-1}}{dx^{m-2n-1}} \frac{1}{\sin x}\right]_{x=C\pi} \tag{40a}$$

$$\sum_{\substack{k=-\infty \\ k\neq -N}}^{\infty} (-1)^k H_{2m-1}^{(2)} (|N+k|A)\ \mathrm{sgn}\,(N+k) = 0 \tag{40b}$$

$$\sum_{\substack{k=-\infty \\ k\neq -N}}^{\infty} (-1)^k H_{2m}^{(2)} (|N+k|A) = 4j(-1)^{n+m}$$

$$\times \sum_{n=1}^{\infty} \frac{A^{2m}}{\sqrt{(2n-1)^2\pi^2 - A^2}[(2n-1)\pi + \sqrt{(2n-1)^2\pi^2 - A^2}]^{2m}}$$

$$+ j(-1)^N \frac{2}{\pi} \sum_{n=0}^{m} \frac{(m+n-1)!(2^{2n-1}-1)B_{2n}(-1)^n}{(m-n)!(2n)!} \left(\frac{2\pi}{A}\right)^{2n} \tag{40c}$$

valid for $m = 1, 2 \ldots, N = 0, \pm 1, \pm 2 \ldots, C \neq 0, \pm 1, \pm 2$ and real, $A > 0$, real and $A \neq \pi, 3\pi, 5\pi \ldots$, and with arg $\{\sqrt{(2n-1)^2\pi^2 - A^2}\} = 0$ or $\pi/2$. Eq. (40a) can be written in the following more convenient form

$$\sum_{k=-\infty}^{\infty} (-1)^k H_m^{(2)}(|C+k|A)[\mathrm{sgn}\,(C+k)]^m$$

$$= 4j \sum_{n=1}^{\infty} \frac{A^m \cos\left[(2n-1)\pi C - m\frac{\pi}{2}\right]}{\sqrt{(2n-1)^2\pi^2 - A^2}[(2n-1)\pi + \sqrt{(2n-1)^2\pi^2 - A^2}]^m}$$

$$+ j\frac{1}{4\pi} \sum_{n=0}^{\leq \frac{m}{2}} \frac{(m-n-1)!}{n!(m-2n-1)!} \left(\frac{2\pi}{A \sin C\pi}\right)^{m-2n} h_{m-2n-1}(\cos C\pi) \tag{41}$$

subject to the same restrictions as those enumerated for Eqs. (40). The polynomials $h_n(u)$ appearing in this equation are defined by

$$h_n(u) = (-1)^n \left[\sin^{n+1} x \frac{d^n}{dx^n} \frac{1}{\sin x}\right]_{\cos x = u} \tag{42}$$

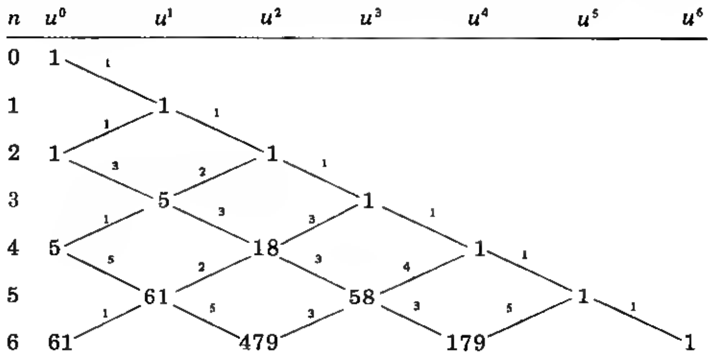
and can be shown to satisfy the following recursion formula

which begins with

$$h_n(u) = nuh_{n-1}(u) + (1-u^2)\frac{dh_{n-1}(u)}{du} \tag{43a}$$

$$h_0(u) = 1 \tag{43b}$$

It appears impossible to give a closed form expression for these polynomials, but their coefficients can easily be calculated by the following scheme which is a consequence of the recursion formula



i.e., it is

$$h_0(u) = 1$$
$$h_1(u) = u$$
$$h_2(u) = 1 + u^2$$
$$h_3(u) = 5u + u^3$$
$$h_4(u) = 5 + 18u^2 + u^4$$

It can be shown, incidentally, that the sum of all coefficients of $h_n(u)$ is equal to $n!$ and that the coefficients of u^0 and u^1 are Euler's numbers. It furthermore appears, but has not been proven, that the coefficients are equal to those in Table 7.2.2 of Ref. 7, p. 260, the "number of permutations of the first N natural numbers with t_u runs up."

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